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FREE HYDROMAGNETIC OSCILLATIONS OF THE EARTH'S CORE AND THE THEORY OF THE GEOMAGNETIC SECULAR VARIATION

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CONTENTS

Free hydromagnetic oscillations of a rotating spherical shell of an incompressible fluid are investigated by means of a simple theoretical model. For each spatial harmonic, rotation gives rise to two distinct modes of oscillation, 'magnetic' and 'inertial', which propagate with different velocities. As an application of the theory, it is shown that if the strength of the toroidal magnetic field in the Earth's core is 100 Oe, then many of the properties of the observed secular changes, including the slow westward drift, of the main geomagnetic field at the Earth's surface can be accounted for in terms of the interaction of magnetic modes in the core with the Earth's poloidal magnetic field. Concomitant magnetic variations due to inertial modes in the core would, owing to their relatively short periods (several days), fail to penetrate to the surface of the Earth, although the eddy currents induced in the lower mantle by these modes might affect the mechanical coupling between the mantle and the core.

1. Introduction

The magnetic field at the surface of the Earth undergoes complicated changes with time. The most rapid changes, occurring on time scales ranging from fractions of a second (subacoustic oscillations) to several days (magnetic storms), have amplitudes less than 1% of the total field and are revealed only by sensitive instruments. They are due to varying electric currents flowing in regions well above the Earth's surface, in the ionosphere and beyond. When these rapid variations, together with longer-period modulations associated with the motion of the sun and of the moon and with the 11 y cycle of solar activity, have been removed from the magnetic record, slow changes, occurring on time scales of decades to centuries, and having amplitudes up to a sizeable fraction of the total field, remain. These changes, long familiar to navigators, constitute the so-called 'geomagnetic secular variation' (g.s.v.).

Spherical harmonic analysis of the annual means of the geomagnetic elements reveals that the g.s.v. originates within the Earth, and that although at the Earth's surface the main geomagnetic field, also of internal origin, is predominantly that of a centred dipole, the time rate of change of the field contains strong quadrupole and higher-order components. This is because the time scale of variations in the dipole field is much longer than that of variations in the non-dipole field. (The *amplitude* of variations in the dipole field is by no means small; according to the data of rock magnetism, the dipole field may have undergone several hundred sudden (geologically speaking) reversals in polarity in the past 5×10^8 y or so!)

It is now generally accepted that the main geomagnetic field arises in the liquid, metallic core of the Earth, the principal argument being simply that the time scale of the g.s.v. is much too short to be accounted for in terms of processes arising in the 'solid' mantle. The pioneering works of Bullard (1949*a*, *b*), Bullard & Gellman (1954), Elsasser (1946*a*, *b*, 1947, 1956) and others on the so-called 'homogeneous dynamo theory' of the dipole field, together with the existence proofs by Backus (1958) and Herzenberg (1958) that such a mechanism is in principle possible, have convinced most geophysicists that the origin of the Earth's magnetism is to be sought in hydromagnetic (magnetohydrodynamic) processes in the core. The next step, therefore, towards a satisfactory explanation of the phenomenon will be to establish a self-consistent theory of the hydromagnetics of the core.

Efforts to provide such a theory are handicapped by the inadequacy of our knowledge of the physical properties and chemical constitution of the Earth's deep interior and by the mathematical difficulties encountered in all realistic studies in fluid dynamics. As with other problems in geophysical fluid dynamics, progress can be made through the formulation and analysis of a hierarchy of highly simplified theoretical models, with a view to interpreting the observational data in terms of basic underlying processes. One such model is treated in the present paper, which examines the properties of small-amplitude hydromagnetic oscillations of a rotating spherical shell of an incompressible fluid about a mean state characterized by a uniform magnetic field which is mainly toroidal. An approximation akin to one introduced by Rossby & Haurwitz (see §§ 3 and 4 below) in the study of planetary-scale tidal oscillations in the atmosphere is employed here to simplify the mathematics without (hopefully) seriously reducing the physical content of the analysis.

In the absence of rotation, the oscillations considered correspond to the superposition of ordinary non-dispersive hydromagnetic waves propagating at the Alfvén speed in both directions along the magnetic lines of force. The effect of rotation is to reduce the phase speed of waves propagating in one of these directions and to increase the phase speed of waves moving in the other direction. Planetary-scale oscillations of the core are so strongly influenced by the Earth's rotation that both types of wave are highly dispersive. The periods of the slow waves are decades or even centuries if the strength of the toroidal magnetic field in the core lies between 50 and 200 Oe (5×10^{-3} to 2×10^{-2} Wb/m^2). Those of the fast waves are of the order of days.

The slow waves have oscillation periods and dispersion times comparable with the time scale of the g.s.v., which suggests that this phenomenon may contain major contributions from the free hydromagnetic oscillations of the core. The electrical conductivity of the mantle, though weak, is sufficient to suppress from the magnetic record at the Earth's surface any manifestation of magnetic variations in the core on the time scale of the fast waves.

The slow waves should move westward relative to the core material; thus, the westward drift of the geomagnetic field relative to the Earth's surface could also be a manifestation of free hydromagnetic oscillations. The high-order spherical harmonic components of the field should drift more rapidly than the low-order harmonics. This type of behaviour is consistent with the observations.

Rikitake's (1956) treatment of hydromagnetic oscillations of a rotating fluid sphere immersed in a uniform magnetic field parallel to the axis of rotation led him to the conclusion that unless the strength of the magnetic field in the core exceeds 10^5 Oe, free hydromagnetic oscillations make no contribution to the g.s.v. The present paper does not support this conclusion. Owing to the mathematical complexity of his model, Rikitake was unable to discuss modes possessing spatial variations with respect to latitude as well as longitude. According to the present work, the modes that he was able to discuss form a special class, on which the influence of an axial magnetic field is atypically weak.

This paper is made up as follows. Section 2 contains general remarks on the hydrodynamics of the Earth's core, emphasizing the role of hydromagnetic forces and of Coriolis forces, and discussing plausible basic states. Small-amplitude hydromagnetic oscillations of a rotating spherical shell of incompressible fluid about the simplest conceivable basic state are analysed in §§ 3, 4 and 5 by means of an idealized theoretical model, and the results thus obtained are applied in § 6 in a discussion of hydromagnetic oscillations of the Earth's core. The principal properties of the g.s.v., considered in the light of § 6, are discussed in § 7. Concluding remarks are made in § 8.

2. General remarks on the hydromagnetics of the Earth's core

(a) *Energetics*

The liquid, metallic core of the Earth occupies a very nearly spherical shell of outer radius 3473 ± 4 km and inner radius 1400 km; the mean radius of the Earth is 6371 km. Below the liquid core is the 'inner core', or 'central body', which is probably solid. Above the core is the solid mantle.

Hydromagnetic flow in the core can in principle be described in terms of the Eulerian hydrodynamical flow velocity, $\mathbf{u}$, and the magnetic field, $\mathbf{B}$, at each point within it and at every instant of time. If the magnetic field in the core is mainly toroidal, with a strength of about 10^{-2} Wb/m^2 (100 Oe) (see § 6 below), the total magnetic energy is 10^{22} J (10^{29} erg), which greatly exceeds the kinetic energy associated with the field of $\mathbf{u}$ (see Elsasser 1956; Hide 1956).

If the mechanism that maintains the magnetic field against Ohmic losses suddenly ceased to function, the field would disappear in about 10^{12} s (3×10^4 y), assuming that σ, the electrical conductivity of the core, is about $3 \times 10^5\,\Omega^{-1}\,\mathrm{m}^{-1}$ (3×10^{-6} e.m.u.) and that the magnetic permeability of the core, μ, is that of free space, namely $4\pi \times 10^{-7}$ henry/m (Bullard & Gellman 1954). Hence, the source of energy responsible for core motions must supply 10^{10} W (10^{17} erg/s) to the system, mainly in the form of magnetic energy. The nature of this source has been discussed by several workers (see Bullard 1949*a*;

Elsasser 1950*a*, *b*; Urey 1952; Hide 1956; Verhoogen 1961; Malkus 1963; Toomre 1966) with interesting but not always conclusive results. Radioactive heating deep within the Earth, gravitational energy released if the Earth is still condensing, and the precessional motion of the Earth may contribute significantly to the stirring of the core.

(*b*) *Hydromagnetic waves*

Transmission of magnetic energy between different parts of the core is probably accomplished mainly by hydromagnetic waves, diffusive processes being entirely unimportant (Roberts 1954; Hide & Roberts 1961). The influence of the Earth's rotation on these waves can be very strong. The nature and extent of this influence may be judged by considering the dispersion relationship for plane hydromagnetic waves in a perfectly conducting, incompressible, inviscid, homogeneous fluid of indefinite extent rotating uniformly with angular velocity $\mathbf{\Omega}$ and immersed in a uniform magnetic field $\mathbf{B}_0$ parallel to $\mathbf{\Omega}$ (see Lehnert 1954; Hide & Roberts 1961, 1962; for the general case when $\mathbf{\Omega}$ and $\mathbf{B}_0$ are not parallel, see Chandrasekhar 1961). If we define the 'Alfvén speed'

$$V_A \equiv (B_0^2/\mu\rho)^{\frac{1}{2}}, \tag{2·1}$$

where ρ is the density of the fluid, and a Rossby number

$$Q \equiv V_A k/2\Omega \tag{2·2}$$

based on the Alfvén speed and on the wavenumber k of the waves, this relationship is

$$\begin{aligned}\omega &= \pm[\Omega \pm (V_A^2 k^2 + \Omega^2)^{\frac{1}{2}}]\\ &= \pm V_A k[(1+4Q^2)^{\frac{1}{2}} \pm 1]/2Q = \pm\Omega[1 \pm (1+4Q^2)^{\frac{1}{2}}],\end{aligned} \tag{2·3}$$

where ω is the angular frequency of the wave. The corresponding phase and group velocities are, respectively,

$$V \equiv \frac{\omega}{k} = \pm\frac{V_A[(1+4Q^2)^{\frac{1}{2}} \pm 1]}{2Q} \tag{2·4}$$

and

$$U \equiv \frac{\mathrm{d}\omega}{\mathrm{d}k} = \pm V_A\left[\frac{2Q}{(1+4Q^2)^{\frac{1}{2}}}\right]. \tag{2·5}$$

According to these equations, Q is the parameter in terms of which the effects of rotation should be measured. When $Q \gg 1$, these effects are insignificant; the waves are then non-dispersive and propagate in either direction along the magnetic lines of force with both phase velocity and group velocity equal to V_A. At the other extreme, when $Q \ll 1$, rotational effects dominate. In discussing the properties of the waves in this case it will be convenient to distinguish between the two pairs of roots of each of the equations (2·3), (2·4) and (2·5) by using the subscript i or m according as the upper or lower sign inside the bracket is taken; when $Q \ll 1$

$$\omega_i = \pm 2\Omega(1+Q^2), \quad V_i = \pm 2\Omega(1+Q^2)/k, \quad U_i = \pm V_A^2 k/\Omega, \tag{2·6}$$

$$\omega_m = \mp V_A^2 k^2/2\Omega, \quad V_m = \mp V_A^2 k/2\Omega, \quad U_m = \mp V_A^2 k/\Omega. \tag{2·7}$$

The first mode will be termed the inertial mode, which, in the absence of a magnetic field (i.e. $Q = 0$), consists of a semidiurnal oscillation (i.e. $\omega_i = \pm 2\Omega$) involving no energy propagation (i.e. $U_i = 0$, see equations (2·6)). The presence of a weak field (i.e. $0 < Q \ll 1$) raises the frequency of the inertial mode and allows energy to propagate slowly along the magnetic lines of force. The frequency and phase velocity of the second mode, the magnetic mode, are much lower than those of the inertial mode. In magnitude the group velocities of the two modes are the same. Both modes are highly dispersive.

Under conditions typical of the core, $V_A = 10^{-1}$ m/s, $k = 2\times10^{-6}\,\mathrm{m}^{-1}$, $\Omega = 7\times10^{-5}$ rad/s and $Q = 1{\cdot}5\times10^{-3}$. Corresponding values of the oscillation period, $2\pi/\omega_m$, the phase velocity, V_m, the group velocity, U_m, and 'dispersion time', $2\pi/k\,|U_m - V_m|$, of the magnetic mode are 2×10^{10} s (700 y), 2×10^{-4} m/s (0·2 mm/s), 4×10^{-4} m/s (0·4 mm/s) and 2×10^{10} s (700 y) respectively. These values suggest that the g.s.v., with its time scale of decades to centuries, may contain significant contributions from free hydromagnetic oscillations of the core and that the detailed theoretical study of these oscillations should be of direct geophysical interest.

(The effect of the Earth's rotation on the oscillation period of the magnetic mode is analogous to the lengthening in the period of a rigid pendulum when its bob contains a gyroscope (see Gray 1918, p. 197). This result bears on Bullard's discussion of the stability of homopolar dynamos; the factor K in his theory (Bullard 1955, p. 759) should be of the order of Q^{-1}, about 10^3 for the core.)

(c) *Basic flow in the core*

It is convenient to separate each of the vectors $\mathbf{u}$ and $\mathbf{B}$ into two parts, the quasi-steady parts, $\mathbf{u}_s$ and $\mathbf{B}_s$, and fluctuating parts, $\mathbf{u}_f$ and $\mathbf{B}_f$, by writing

$$\mathbf{u} = \mathbf{u}_s + \mathbf{u}_f, \quad \mathbf{B} = \mathbf{B}_s + \mathbf{B}_f. \tag{2·8}$$

We have no direct knowledge of $\mathbf{u}$ and $\mathbf{B}$, and what indirect knowledge we now possess, mainly through efforts to find a geophysically plausible dynamo mechanism, leaves much to be desired. However, even if $\mathbf{u}_s$ and $\mathbf{B}_s$ were known exactly it is unlikely that they would be sufficiently simple in form to be of any use in a discussion of free oscillations.

In a rapidly rotating fluid only quite weak meridional motions are required to build up a strong zonal shear; consequently one property of $\mathbf{u}_s$ is that it is probably mainly zonal. Hide (1956, 1960 *a*, *b*) has discussed the principal factors influencing the dependence of $\overline{V}$, the zonal component of $\mathbf{u}_s$, on r and Z in a rotating spherical shell of fluid of non-uniform density, where (r, λ, Z) are cylindrical polar coordinates (see figure 1). These factors are the angular speed of rotation, Ω, and the vertical gradient of potential density, Γ (say), as measured by the so-called Brunt–Väisälä frequency, Ω_B (see Eckart 1960), where

$$\Omega_B^2 = -|\mathbf{g}|\,\Gamma/\rho$$

if $\mathbf{g}$ is the acceleration of gravity and ρ is the actual density. When Ω_B is real and much greater than Ω (which is usually the case in the atmosphere and in the oceans) the motions are constrained by gravity to follow nearly spherical surfaces. When, on the other hand, $|\Omega_B| \ll \Omega$, the Proudman–Taylor theorem for quasi-steady motions,

$$\partial\mathbf{u}/\partial z = 0, \tag{2·9}$$

applies (see Hide 1956).

Although the value of Ω_B for the Earth's core is not known precisely, it should not exceed that corresponding to an isothermal core, which is much less than Ω. Hence equation (2·9) will roughly apply, in which case $\partial\overline{V}/\partial Z$ is so small that, for the purpose of the present discussion, a good approximation to $\mathbf{u}_s$ should be

$$\mathbf{u}_s = (0, \overline{V}, 0); \quad \overline{V} = \overline{V}(r). \tag{2·10}$$

This amounts to a law of iso-rotation for concentric cylindrical shells.

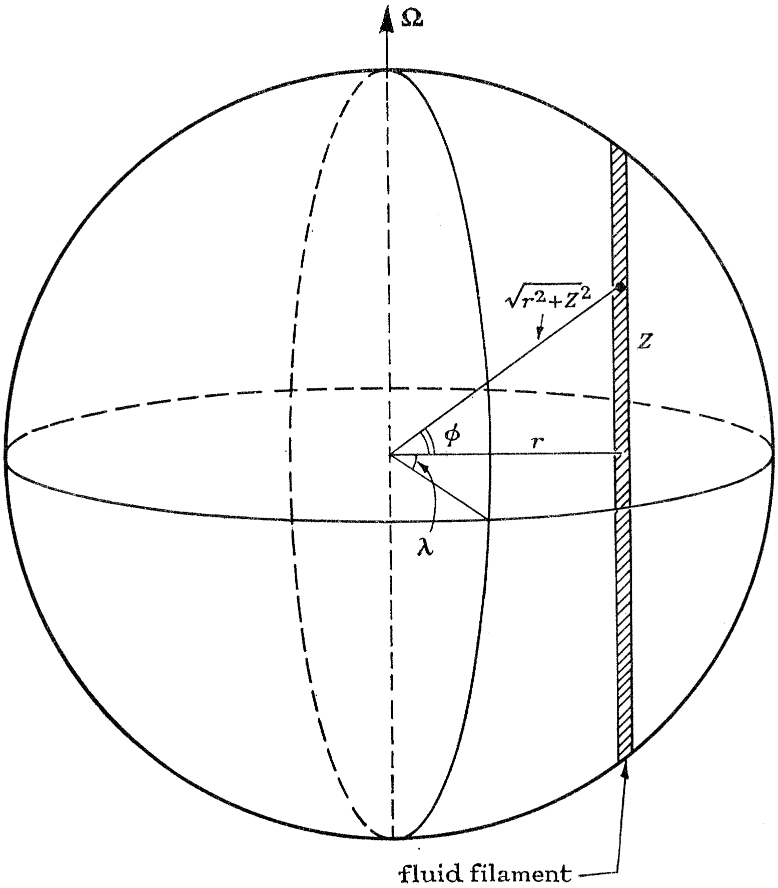

FIGURE 1

The motion specified by equation (2·10) need not be inconsistent with the requirement that on the average no net torque is applied by the core on the mantle. The mechanical coupling between the core and mantle is probably due not to viscous friction, acting locally at the core-mantle interface, but to electric currents leaking out of the core into the lower mantle and interacting with the magnetic field present in that region (see Bullard & Gellman 1954, also § 8 below).

(The consequences of equation (2·9) are often quite dramatic (see Taylor 1923; also Hide 1960*a*, *b*, 1961; Hide & Ibbetson 1966). This equation implies strong hydrodynamical coupling between remote parts of the fluid and raises a question recently put to me by Dr E. H. Vestine: how smooth must the core/mantle interface be for the topography of this interface to produce no significant effect on core motions, and hence, presumably, on the Earth's main magnetic field? While this question cannot be answered with certainty, simple arguments, given in a different context (Hide 1961, 1963), suggest that the hydrodynamical effects of quite shallow planetary-scale features, having vertical dimensions of only a kilometre or so, could penetrate to great depths within the core. Although the geomagnetic field may possess features associated with horizontal variations in the properties of the lower mantle (see §7), this is not yet certain. Studies of these and related questions are of general geophysical interest, and might shed light on mechanical processes in the lower mantle, especially if they lead to an estimate of the vertical extent of mantle convection.)

A systematic discussion of the dynamics of the basic velocity field $\mathbf{u}_s$ has not yet been given. In connexion with what has been done (see, for instance, Frenkel 1945; Bullard 1949*a*; Elsasser 1950*a*, *b*, 1956; Bullard & Gellman 1954; Runcorn 1954; Parker 1955; Inglis 1955; Hide 1953, 1956; Hide & Roberts 1961; Taylor 1963; Malkus 1963; Toomre 1966), suffice it to remark here that since energy dissipation and angular momentum transfer are due mainly to hydromagnetic effects and not to molecular viscosity, a close relation must exist between $\mathbf{u}_s$ and $\mathbf{B}_s$ (see equations (2·8) and 2·10)).

The spatial properties of $\mathbf{B}$ are best discussed by first dividing it into its poloidal and toroidal parts, $\mathbf{B}_p$ and $\mathbf{B}_T$ respectively (see Bullard & Gellman 1954). Only $\mathbf{B}_p$ has a radial component, with lines of force passing out of the core, through the mantle, and on into space. The toroidal magnetic field, $\mathbf{B}_T$, if it exists, has no radial component and is thus confined to its region of origin, the core. Extrapolation of the field at the Earth's surface leads (in principle) to information about $\mathbf{B}_p$ at the core-mantle interface. $\mathbf{B}_T$, on the other hand, cannot be detected directly (see, however, Runcorn 1954; Roberts & Lowes 1961).

We can regard $\mathbf{B}_T$ as the result of the interaction of $\mathbf{B}_p$ with zonal shearing motions (see equation (2·10)), the ratio $\langle B_T\rangle \div \langle B_p\rangle$ being roughly of the order of a magnetic Reynolds number (see Bullard & Gellman 1954; Hide & Roberts 1961), namely

$$G = 2\langle \overline{V}\rangle H\sigma\mu, \tag{2·11}$$

if $\langle B_T\rangle$, $\langle B_p\rangle$ and $\langle \overline{V}\rangle$ denote spatial averages (regardless of sign) of the magnitude of $\mathbf{B}_T$, $\mathbf{B}_p$ and $\overline{V}(r)$ respectively and $2H$ is the thickness of the core (see figure 3, §3 below). Hence, if $(\mu\sigma)^{-1} \sim 3\,\mathrm{m^2/s}$, $2H \sim 2\times 10^6\,\mathrm{m}$, and $\langle B_p\rangle \sim 5\,\mathrm{Oe}$ $(5\times 10^{-4}\,\mathrm{Wb/m^2})$, then

$$G \sim 6\times 10^6 \langle \overline{V}\rangle, \quad \langle B_T\rangle \sim 3\times 10^7 \langle \overline{V}\rangle\,\mathrm{Oe} = 3000\langle \overline{V}\rangle\,(\mathrm{Wb/m^2}), \tag{2·12a, b}$$

if $\langle \overline{V}\rangle$ is measured in metres per second. The numerical factors in equations (2·12) are probably uncertain by as much as an order of magnitude.

While the speed of motion relative to the Earth's surface of the Earth's non-dipole field suggests that $\langle \overline{V}\rangle$ is unlikely to exceed approximately $10^{-3}\,\mathrm{m/s}$ (1 mm/s), in the absence of an acceptable theory of the g.s.v. estimates of $\langle \overline{V}\rangle$ will remain uncertain. If $\langle \overline{V}\rangle \sim 10^{-3}\,\mathrm{m/s}$, then, by equations (2·12), $G \sim 6000$ and $\langle B_T\rangle \sim 3\times 10^4\,\mathrm{Oe}$. On the other hand, if, as is strongly suggested by the present work (see §§ 6 and 7 below), the principal properties of the g.s.v., including the westward drift, are manifestations of free oscillations of the Earth's core, then $\langle B_T\rangle \sim 100\,\mathrm{Oe}$, and by equations (2·12), $G \sim 20$, $\langle \overline{V}\rangle \sim 3\times 10^{-6}\,\mathrm{m/s}$ $(3\times 10^{-3}\,\mathrm{mm/s})$.

In spite of these uncertainties it is very likely that $\langle B_T\rangle \gg \langle B_p\rangle$, and this is one of the assumptions we shall make when we formulate the theoretical model discussed in § 5 below.

The problem of discussing in general terms all possible modes of hydromagnetic oscillation of the core would be quite complex, even if $\mathbf{u}_s$ and $\mathbf{B}_s$ (see equations (2·8)) were very simple in form. The difficulties are mathematical and stem from the need to work with spherical coordinates. They can be reduced by finding an equivalent set of plane coordinates which retains only those effects due to the spherical shape of the boundaries that are qualitatively significant. Such a set of coordinates is suggested by theoretical work on long-period oscillations of the atmosphere and oceans.

3. Simplified system of coordinates

Tidal oscillations of a fluid layer on a rotating earth are of two kinds—the so-called Laplace oscillations of the first and second class (Hough 1898; Lamb 1945). In the limit of zero rotation first-class oscillations become ordinary long gravity waves and second-class oscillations become quasi-horizontal steady motions. The frequency of both types of oscillation increases with increasing Ω. The second-class oscillations correspond to the inertial modes; the maximum possible angular frequency of a pure inertial mode is 2Ω.

Some years ago Rossby (1939) introduced the so-called 'β-plane' method of treating inertial oscillations of period much greater than $2\pi/\Omega$ in a thin rotating spherical shell of fluid of mean radius R in regions remote from the equator. Motions possessing no radial component (toroidal motions) are treated in an approximate fashion by using a local Cartesian frame whose origin is located at the mean latitude ϕ_0 of the region under consideration, the x, y and z axes being directed eastward, northward and upward respectively, and, so far as the horizontal Coriolis accelerations are concerned, taking $\Omega \sin\phi$, the vertical component of $\mathbf{\Omega}$, as the angular velocity of rotation, where ϕ is the angle of latitude (see figure 1). The variation of $\Omega\sin\phi$ with ϕ is crucial and cannot be ignored. A further simplification is introduced by expressing the Coriolis parameter, $f \equiv 2\Omega\sin\phi$, in the form

$$f = f_0 + \beta y, \tag{3·1}$$

where
$$f_0 = 2\Omega\sin\phi_0, \quad \beta = 2\Omega\cos\phi_0/R \tag{3·2}$$

(cf. equation (3·6) below), and treating f_0 and β as constants so that $\mathrm{d}f/\mathrm{d}y = \beta$.

Rossby thus eliminated many mathematical difficulties and was able to isolate what are now called Rossby or Rossby–Haurwitz waves (see §4 below); Rossby called them 'planetary waves'. Several workers subsequently recognized the equivalence of Rossby waves to certain modes of second-class oscillations in tidal theory (Haurwitz 1940; Arons & Stommel 1956; Eckart 1960; see also Phillips 1963; Longuet-Higgins 1964).

The β-plane method, when combined with physical insight and not pressed too far, has proved very fruitful in dynamical meteorology and oceanography. Although it cannot be taken over directly in the treatment of thick spherical shells of fluid, by investigating the physics of the β-plane method it is possible to formulate a comparable method which might be applicable to the Earth's core.

We have already seen that slow steady motions of an inviscid homogeneous incompressible fluid which rotates rapidly about the Z axis will be two-dimensional, in planes perpendicular to the axis of rotation (see equation (2·9)). Consider the motion of a filament of fluid that everywhere lies parallel to the axis of rotation (see figure 2) and ignore for the moment hydromagnetic forces. It may be shown that if $H \ll R$ then

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta + 2\Omega}{2H/\sin\phi}\right) = 0, \tag{3·3}$$

where ζ is the Z component of vorticity, namely $r^{-1}[\partial(ru_\lambda)/\partial r - \partial u_r/\partial\lambda]$, u_r and u_λ being the r and λ components of velocity of the filament, and $\mathrm{D}/\mathrm{D}t$ is the derivative with respect to time following the motion of the filament (see Veronis 1963; Hide 1960*b*). According to this equation, if the filament changes its latitude, ζ changes by an amount proportional to the concomitant change in length of the filament. This is in accordance with Kelvin's theorem that, in the absence of friction and density inhomogeneities, the circulation around the axis of the filament must be conserved. A filament moving poleward decreases in length and—as we are considering an incompressible fluid—its cross-sectional area increases. Since the circulation is proportional to the product of $(2\Omega+\zeta)$ and the cross-sectional area, conservation of circulation demands a reduction in ζ. By the same reasoning, an equatorward-moving filament *increases* its relative vorticity.

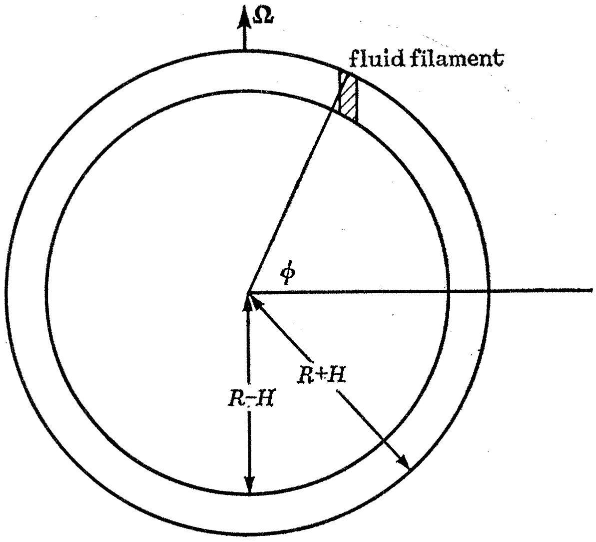

FIGURE 2

It is readily demonstrated that equation (3·3) leads directly to Rossby's β-plane equation of motion (see, for instance, Veronis 1963). If $\beta = 0$ (see equation (3·1)) in Rossby's representation, individual fluid particles conserve their relative vorticity as they move. Thus, the term βy on the right-hand side of equation (3·1) takes into account the spherical shape of the boundaries.

Now consider the motion of a filament in a thick spherical shell (see figure 3). In latitudes greater than ϕ^*, where

$$\phi^* = \cos^{-1}[(R-H)/(R+H)] \tag{3·4}$$

($\phi^* = 68^\circ \pm 2^\circ$ for the Earth's core; see Hide 1960*a*), the effect of the spherical boundaries is qualitatively similar to that expressed by equation (3·2), although the expression for the length of the filament is now a little more complicated than $2H/\sin\phi$. On the other hand, at lower latitudes, i.e. at $\phi < \phi^*$, in place of equation (3·3) we have

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\frac{\zeta+2\Omega}{2(R+H)\sin\phi}\right] = 0, \tag{3·5}$$

with the same physical interpretation in terms of Kelvin's theorem. In contrast to the former case, an increase in latitude of the ends of the filament brings about an *increase* rather than a decrease of ζ, and vice versa.

The preceding discussion suggests that the β-plane treatment of large-scale inertial modes in a thick homogeneous shell of fluid might constitute a good first approximation to an accurate analysis, provided that in equation (3·1) the sign of β, the factor in the term representing the variation of f with latitude, is negative. In place of equation (3·2) we shall take

$$f_0 = 2\Omega \sin \phi_0, \quad \beta = -2\Omega \cos \phi_0/(R+H); \tag{3·6}$$

when applied to the Earth's interior Ω is the angular speed of rotation of the Earth, $(R+H)$ is the outer radius of the liquid core (see figure 3) and $2H$ is the thickness of the core.

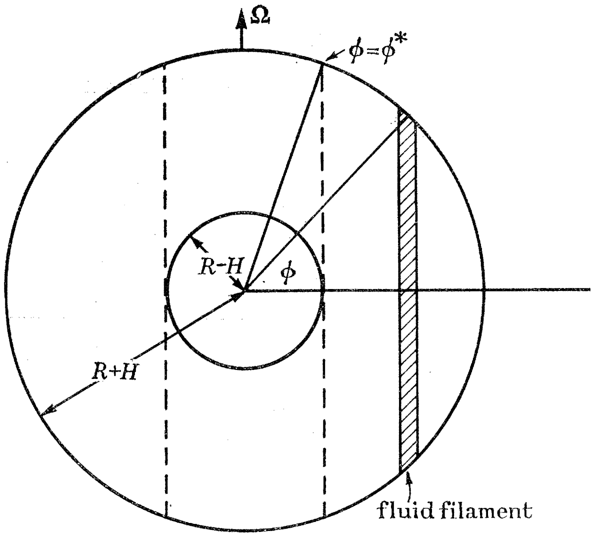

FIGURE 3

As the β-plane method is essentially a local treatment, there is some arbitrariness in applying it to find the normal modes of a spherical system. The main value of the method lies in its mathematical simplicity. Any results obtained by its use should be regarded as suggestive of what to look for in a more accurate analysis which would take the spherical geometry fully into account.

If in any particular problem the β-plane method gives qualitatively sound results, a hypothetical latitude angle ϕ_1 can be defined such that when $\phi_0 = \phi_1$ (see equations (3·1), (3·2) and (3·3)) the quantitative agreement of the β-plane results with those of a full theoretical treatment is better than at any other value of ϕ_0. In the absence of a measure of ϕ_1, this quantity will have to be guessed. Although unrepresentative conditions obtain near $\phi = 0$ (the equator) and poleward of $\phi = \phi^* = 68° \pm 2°$ (see equation (3·4)), equation (3·5) should have some validity at other latitudes. Since more than eight-tenths of the volume of the core lies equatorward of latitude 68°, ϕ_1 should be less than 68°. One could be naïve and suggest that in the absence of a better estimate $\phi_1 = \frac{1}{2}\phi^* = 34°$, in which case

$$f_0 = 0·75 \times 10^{-4}\,\mathrm{s}^{-1}, \quad \beta = -3·4 \times 10^{-13}\,\mathrm{cm}^{-1}\,\mathrm{s}^{-1}$$

(cf. equation (3·6)). It would be very surprising, of course, if ϕ_1 turned out to be 34°; perhaps the best we can guess is $50° < \phi_1 < 20°$. Fortunately, uncertainties in ϕ_1 affect none of the conclusions of this paper.

4. Inertial oscillations in the absence of a magnetic field

Before discussing hydromagnetic oscillations, it will be instructive to outline the β-plane treatment of the inertial oscillations corresponding to the Laplace tidal oscillations of the second class (Rossby–Haurwitz waves). The equations of two-dimensional motion of an inviscid, incompressible fluid are:

$$\frac{Du}{Dt} - fv = -\frac{\partial P}{\partial x}, \tag{4·1}$$

$$\frac{Dv}{Dt} + fu = -\frac{\partial P}{\partial y}, \tag{4·2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4·3}$$

where u and v denote, respectively, the components, relative to the rotating frame of reference, of the Eulerian velocity vector, $\mathbf{u}$, in the x (eastward) and y (northward) directions, D/Dt is the total derivative $(\partial/\partial t + u\partial/\partial x + v\,\partial/\partial y)$, P is the dynamic pressure divided by ρ, the density (assumed uniform) and f $(=f_0+\beta y$, see equations (3·1), (3·2) and (3·6)) is the Coriolis parameter.

Eliminate P between equations (4·1) and (4·2), making use of equations (3·1) and (4·3), and show that the relative vorticity,

$$\zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{4·4}$$

satisfies the equation

$$\frac{D\zeta}{Dt} + v\beta = 0. \tag{4·5}$$

Operate on this equation with $\nabla_H^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ and make use of equations (4·3) and (4·4) to find

$$\frac{D}{Dt}\nabla_H^2\zeta + \beta\frac{\partial \zeta}{\partial x} = 0. \tag{4·6}$$

Now consider the behaviour of a small-amplitude disturbance superimposed on a uniform basic flow in the x direction. Let U_0 be the speed of this flow relative to the rotating frame, (u_1, v_1) the components of the disturbance velocity and ζ_1 the vorticity of the disturbance. Since (by hypothesis) $\mathbf{U}_0$ has zero vorticity, on neglecting second and higher-order terms in u_1, v_1 and ζ_1 equations (4·3), (4·4), (4·5) and (4·6) lead to

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \quad \zeta_1 = \frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y}, \quad \frac{d\zeta_1}{dt} + v_1\beta = 0 \tag{4·7 a–c}$$

and

$$\frac{d}{dt}\nabla_H^2\zeta_1 + \beta\frac{\partial \zeta_1}{\partial x} = 0, \tag{4·8}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x}. \tag{4·9}$$

If the disturbance varies harmonically in space with wavelengths $2\pi/k$ and $2\pi/l$ in the x and y directions respectively, and in time with angular frequency $\hat{\omega}$, we can write

$$\{u_1, v_1, \zeta_1\} = \{\bar{u}, \bar{v}, \bar{\zeta}\}\exp i(kx + ly - \hat{\omega}t), \tag{4·10}$$

where $\bar{u}$, $\bar{v}$ and $\bar{\zeta}$ are amplitude factors. We shall treat k and l as real and positive quantities. As $\hat{\omega}$ is always real (see equation (4·14)), undamped oscillatory motion occurs for all k and l. Positive $\hat{\omega}$ corresponds to eastward phase propagation and negative $\hat{\omega}$ to westward phase propagation.

Now substitute equation (4·10) in equations (4·7) and find

$$k\bar{u}+l\bar{v}=0, \quad \bar{\zeta}=\mathrm{i}(k\bar{v}-l\bar{u}), \quad -\mathrm{i}\omega\bar{\zeta}+\bar{v}\beta=0, \tag{4·11 a–c}$$

where
$$\omega\equiv\hat{\omega}-U_0k. \tag{4·12}$$

According to equations (4·11 a, b), (4·8), (4·9) and (4·10)

$$\bar{u}=\mathrm{i}l\bar{\zeta}/(k^2+l^2), \quad \bar{v}=-\mathrm{i}k\bar{\zeta}/(k^2+l^2), \tag{4·13}$$

and
$$\omega\equiv\hat{\omega}-U_0k=-\beta k/(k^2+l^2). \tag{4·14}$$

Equation (4·14), first obtained by Haurwitz (1940) in his generalization of Rossby's work (1939) to the case $l \neq 0$, is the required dispersion relation.

The phase velocity, V, of the waves in the positive x direction relative to the basic flow is given by
$$V\equiv\omega/k=-\beta/(k^2+l^2) \tag{4·15}$$
(see equations (4·12) and (4·14)), whose sense is westward or eastward according as β is positive or negative (see equations (3·2) and (3·6)). The corresponding group velocity, U, relative to the basic flow of these highly dispersive waves is given by

$$U\equiv\partial\omega/\partial k=\beta(k^2-l^2)/(k^2+l^2)^2, \tag{4·16}$$

which, when $k>l$, is eastward or westward according as β is positive or negative, and vice versa when $k<l$. A disturbance for which $(V+U_0)=0$, i.e. $U_0(k^2+l^2)=\beta$, is stationary relative to the rotating frame, although the corresponding group velocity relative to the rotating frame, (U_0+U), is then $2\beta k^2/(k^2+l^2)^2$, which does not vanish.

The physical interpretation of equation (4·15) is quite straightforward. According to equation (4·5), when β is positive filaments displaced poleward acquire negative relative vorticity while those displaced equatorward acquire positive vorticity. Mutual interaction of these vortices produces a westward motion of the whole wave pattern. When β is negative the vorticity changes are reversed and so is the sense of phase propagation.

5. Effect of a magnetic field on inertial oscillations

(a) *Basic equations*

Now consider the effect of a magnetic field on the inertial oscillations discussed in § 4. Assume that the fluid is a perfect conductor of electricity†; in considering free oscillations of the core this is probably a valid assumption, since the periods of these oscillations (see § 6 below) turn out to be less, and usually much less, than the corresponding time scales of free decay (see Hide 1956; Hide & Roberts 1961, 1962; see also § 2 above).

A magnetic field perpendicular to the (x, y) plane has no effect on two-dimensional oscillations (cf. equation (5·10) below); hence, without loss of generality we can restrict attention to the case of a magnetic field whose lines of force are parallel to the (x, y) plane.

† Because $\mu\sigma\nu$ (where ν denotes kinematic viscosity) is probably much less than unity for the core, at the core-mantle interface we must require as a boundary condition on the hydromagnetic equations for an ideal fluid that **B** should be continuous; **u** is not required to satisfy the no-slip condition (Stewartson 1960; Hide & Roberts 1962).

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Moreover, we shall treat the case of an impressed magnetic field of uniform strength B_0 making an angle θ with the x axis. Although this idealization, together with the assumption that the basic relative flow is uniform, leads to results suggestive of geophysical interest (see below), a more realistic analysis should treat the basic magnetic and velocity fields as non-uniform, recognizing that the electric currents responsible for the toroidal field in the core flow mainly within the core, and that there is an intimate relation between this toroidal field and the shear of the basic flow (cf. equations (2·10), (2·11) and (2·12)).

The equations of two-dimensional hydromagnetic flow of an inviscid fluid can be obtained by adding to equations (4·1) and (4·2) terms representing the force per unit mass, $\rho^{-1}(\mathbf{j}\times\mathbf{B})$, where $\mathbf{j}$, components (j_x, j_y, j_z), is the density of the electric current flowing in the system and $\mathbf{B}$, components (B_x, B_y, B_z), is the total magnetic field, i.e. the impressed field plus the contribution due to motional induction (see any text on hydromagnetics such as Cowling 1957; Hide & Roberts 1962; Alfvén & Fälthammar 1963). In virtue of Ampère's law,

$$\nabla\times\mathbf{B} = \mu\mathbf{j}, \tag{5·1}$$

the x and y components of $\mathbf{j}\times\mathbf{B}$ are

$$\mu^{-1}\left\{-\frac{1}{2}\frac{\partial B_z^2}{\partial x}-B_y\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)\right\} \quad\text{and}\quad \mu^{-1}\left\{-\frac{1}{2}\frac{\partial B_z^2}{\partial y}+B_x\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)\right\},$$

respectively, so that equations (4·1) and (4·2) become

$$\frac{\mathrm{D}u}{\mathrm{D}t}-fv = -\frac{\partial}{\partial x}\left(P+\frac{B_z^2}{2\mu\rho}\right)-\frac{B_y}{\mu\rho}\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right), \tag{5·2}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t}+fu = -\frac{\partial}{\partial y}\left(P+\frac{B_z^2}{2\mu\rho}\right)+\frac{B_x}{\mu\rho}\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right). \tag{5·3}$$

The continuity equation, (4·3), is, of course, unchanged. The magnetic field $\mathbf{B}$ satisfies a similar equation,

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y} = 0, \tag{5·4}$$

since $\mathbf{B}$ is a solenoidal vector and we are considering a two-dimensional system in which variations with respect to z are ignored.

Because the fluid is assumed to have perfect conductivity, the effective electric field experienced by a moving fluid particle must vanish; otherwise, by Ohm's law, electric currents of infinite strength, producing infinite magnetic fields, would flow (see equation (5·1)). Hence

$$\mathbf{E}+\mathbf{u}\times\mathbf{B} = 0, \tag{5·5}$$

where $\mathbf{E}$ is the electric field measured in the basic rotating frame of reference.

$\mathbf{E}$ satisfies Faraday's law of induction

$$\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t, \tag{5·6}$$

which, when combined with equation (5·5), leads to

$$\partial\mathbf{B}/\partial t = \nabla\times(\mathbf{u}\times\mathbf{B}) \tag{5·7}$$

Hence, to complete our set of equations, to equations (4·3), (3·9), (5·2), (5·3) and (5·4) must be added the following equations, obtained by combining equation (5·7) with equations (4·3) and (5·4):

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)B_x=\left(B_x\frac{\partial}{\partial x}+B_y\frac{\partial}{\partial y}\right)u, \tag{5·8}$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)B_y=\left(B_x\frac{\partial}{\partial x}+B_y\frac{\partial}{\partial y}\right)v, \tag{5·9}$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)B_z=0. \tag{5·10}$$

Eliminate $(P+B_z^2/2\mu\rho)$ between equations (5·2) and (5·3) and thus find a modified vorticity equation (cf. equation (4·5)), namely:

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t}+v\beta=\frac{1}{\mu\rho}\left(B_x\frac{\partial}{\partial x}+B_y\frac{\partial}{\partial y}\right)\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right). \tag{5·11}$$

(This equation may be obtained directly from a more general expression (Hide, unpublished) corresponding to the extension to the hydromagnetic case of Ertel's potential vorticity theorem (see Eliassen & Kleinschmidt 1957).)

Operate on this equation with $\mathrm{D}/\mathrm{D}t$ and make use of equations (5·8) and (5·9) to find

$$\frac{\mathrm{D}^2\zeta}{\mathrm{D}t^2}+\beta\frac{\mathrm{D}v}{\mathrm{D}t}-\frac{1}{\mu\rho}\left(B_x\frac{\partial}{\partial x}+B_y\frac{\partial}{\partial y}\right)^2\zeta=0 \tag{5·12}$$

(cf. equation (4·5)). Finally, eliminate v between equations (5·12), (4·3) and (4·4) and find

$$\left\{\frac{\mathrm{D}^2}{\mathrm{D}t^2}-\frac{1}{\mu\rho}\left(B_x\frac{\partial}{\partial x}+B_y\frac{\partial}{\partial y}\right)^2\right\}\nabla_H^2\zeta+\beta\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial\zeta}{\partial x}=0 \tag{5·13}$$

(cf. equation (4·6)).

(*b*) *Small-amplitude oscillations*

Now consider the behaviour of a small-amplitude disturbance, velocity components (u_1, v_1) and magnetic field components (b_x, b_y, b_z), superimposed on a uniform flow with relative velocity U_0 in the x direction and a uniform magnetic field of strength B_0 inclined at an angle θ to the x direction.

By equation (5·1), the assumption that B_0 is uniform implies that, in the basic state, no electric current flows in the fluid. However, by equation (5·5) a uniform electric field of strength $U_0 B_0 \sin\theta$ perpendicular to the x, y plane will be present.

(The presence of this electric field is only consistent with the model chosen if the bounding surfaces of the fluid are electrically isolated from one another outside the fluid. Otherwise an electric current would flow in the basic state, invalidating the assumption that $\mathbf{B}_0$ is uniform (see equation (5·6)). As already noted, the present work should be extended in the future to more realistic cases in which the basic magnetic field and relative velocity field are not irrotational.)

Since the amplitude of the disturbance is small, we can neglect second and higher-order quantities in u_1, v_1, etc. The equations governing the disturbance are (4·7 *a*, *b*) together with

$$\frac{\partial b_x}{\partial x}+\frac{\partial b_y}{\partial y}=0 \tag{5·14}$$

(see equation (5·4)),

$$\frac{\mathrm{d}b_x}{\mathrm{d}t} = B_0\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right)u, \tag{5·15}$$

$$\frac{\mathrm{d}b_y}{\mathrm{d}t} = B_0\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right)v, \tag{5·16}$$

$$\frac{\mathrm{d}b_z}{\mathrm{d}t} = 0 \tag{5·17}$$

(see equations (5·8), (5·9) and (5·10)),

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2} - V_A^2\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right)^2\right\}\zeta_1+\beta\frac{\mathrm{d}v_1}{\mathrm{d}t} = 0 \tag{5·18}$$

(see equation (5·12)) and

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2} - V_A^2\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right)^2\right\}\nabla_H^2\zeta_1+\beta\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\zeta_1}{\partial x} = 0 \tag{5·19}$$

(see equation (5·13)), where $\mathrm{d}/\mathrm{d}t$ is defined by equation (4·9) and V_A is the Alfvén speed based on the impressed magnetic field (see equation (2·1)). Observe that b_z enters only one of these equations, (5·17), so that in treating the dynamics of the system we can ignore b_z. According to equation (5·10), any component of $\mathbf{B}$ perpendicular to the x, y plane is simply advected with the motion, without contributing to the hydromagnetic forces acting.

As before, consider a disturbance that varies harmonically in space and time, writing

$$\{u_1, v_1, \zeta_1, b_x, b_y\} = \{\bar{u}, \bar{v}, \bar{\zeta}, \bar{b}_x, \bar{b}_y\}\exp \mathrm{i}(kx+ly-\hat{\omega}t) \tag{5·20}$$

(cf. equation (4·10)), which, when substituted into equations (4·7), (5·14), (5·15), (5·16), (5·18) and (5·19), lead to the following results:

$$\bar{u} = \mathrm{i}l\bar{\zeta}/(k^2+l^2), \quad \bar{v} = -\mathrm{i}k\bar{\zeta}/(k^2+l^2) \tag{5·21}$$

(cf. equation (4·13)),

$$\bar{b}_x = -\frac{\mathrm{i}B_0\kappa l\bar{\zeta}}{\omega(k^2+l^2)}, \quad \bar{b}_y = \frac{\mathrm{i}B_0\kappa k\bar{\zeta}}{\omega(k^2+l^2)} \tag{5·22}$$

and

$$\omega^2+\beta k\omega/(k^2+l^2)-V_A^2\kappa^2 = 0 \tag{5·23}$$

(cf. equation (4·14)), where

$$\kappa \equiv k\cos\theta + l\sin\theta, \tag{5·24}$$

recalling that $\omega \equiv \hat{\omega} - U_0 k$ (see equation (4·12)).

Equation (5·23) is the required dispersion relationship. If we introduce a characteristic wavenumber based on V_A and β (see equations (2·1) and (3·1)), namely

$$k_0 \equiv (|\beta|/2V_A)^{\frac{1}{2}}, \tag{5·25}$$

and a dimensionless parameter

$$\gamma \equiv \left|\frac{\kappa(k^2+l^2)}{kk_0^2}\right| = \left|\frac{2V_A(k^2+l^2)\,(k\cos\theta+l\sin\theta)}{k\beta}\right| \tag{5·26}$$

(see equations (5·24) and (5·25)), then equation (5·23) can be written in two equivalent forms:

$$\omega^2+2\,\mathrm{sgn}\,(\beta)\,V_A|\kappa|\gamma^{-1}\omega - V_A^2\kappa^2 = 0 \tag{5·27}$$

and

$$\omega^2+\frac{\beta k\omega}{k^2+l^2}-\frac{\beta^2k^2\gamma^2}{4(k^2+l^2)^2} = 0 \tag{5·28}$$

(cf. equation (4·14)). For given wavenumbers, k and l, and angle, θ, made by the impressed uniform magnetic field of strength B_0 to the west–east (x) direction, γ may be regarded either as a measure of the Alfvén speed V_A (proportional to B_0, see equation (2·1)) in terms of β, the variation of Coriolis parameter with latitude, or, alternatively, as an inverse measure of β in terms of V_A, in which case γ has some of the characteristics of a Rossby number (see equation (2·2)). If V_A and β are regarded as fixed, γ varies only with k, l and θ, and if l and θ are also regarded as fixed, then γ is a measure of k, the east–west wavenumber. This measure, expressed by

$$|(k\cos\theta + l\sin\theta)\,(k^2+l^2)/k|^{\frac{1}{2}} = \gamma^{\frac{1}{2}}k_0 \tag{5·29}$$

(see equations (5·25) and (5·26)), though complicated in general, reduces to

$$\gamma^{\frac{1}{2}} = k/k_0, \tag{5·30}$$

when $l = \theta = 0$.

The two roots, ω_i and ω_m (say), of equation (5·23) are:

$$\omega_i, \omega_m = -\operatorname{sgn}(\beta)\, V_A|\kappa|\,\{\pm(1+\gamma^{-2})^{\frac{1}{2}}+\gamma^{-1}\} \tag{5·31}$$

(see equation (5·27)), equivalent to

$$\omega_i, \omega_m = -\frac{\beta k}{2(k^2+l^2)}\{1\pm(1+\gamma^2)^{\frac{1}{2}}\} \tag{5·32}$$

(see equation (5·28)), according as the upper or lower sign is taken. These roots have their respective counterparts in the quantities ω_i and ω_m for the inertial and magnetic modes of oscillation of plane waves in an infinite medium (see equations (2·6) and (2·7)), whence the notation. When $\gamma = 0$, ω_i is equal to the Rossby–Haurwitz value (cf. equations (4·14) and (5·32)) and ω_m is zero.

For each spatial harmonic, characterized by a combination of definite real and positive values of k and l, there are two different frequencies, ω_i and ω_m. Any general disturbance of small amplitude can, in virtue of Fourier's theorem, be synthesized in terms of a superposition of spatial harmonics, and will contain many modes of oscillation, both inertial and magnetic.

It follows immediately from equation (5·31) or (5·32) that since γ, V_A, k and l are essentially positive quantities, the signs of ω_i and ω_m, and hence of $V_i \equiv \omega_i/k$ and $V_m \equiv \omega_m/k$, the speeds of phase propagation in the x direction for the two modes, depend only on the sign of β; thus

$$\operatorname{sgn}(V_i) = -\operatorname{sgn}(\beta), \quad \operatorname{sgn}(V_m) = \operatorname{sgn}(\beta). \tag{5·33}$$

Phase propagation for the inertial mode is westward or eastward according as $\beta \gtrless 0$ and vice versa for the magnetic mode.

A further immediate consequence of equation (5·31) or (5·32) is that

$$|\omega_i| \geqslant V_A|\kappa| \geqslant |\omega_m|, \tag{5·34}$$

the equality signs holding when $\gamma^{-1} = 0$ (i.e. infinite magnetic field strength or zero rotation); the effect of rotation is always to increase the magnitude of the frequency of the inertial mode and to reduce that of the magnetic mode.

(c) *Some limiting cases*

The dependence of the properties of the inertial and magnetic modes on the parameter γ is best illustrated by considering the expressions for ω_i and ω_m in the limiting cases $\gamma \gg 1$ (i.e. strong magnetic field or slow rotation) and $\gamma \ll 1$ (i.e. weak magnetic field or rapid rotation). These expressions, together with those for an intermediate case, $\gamma = 1$, are listed in table 1.

TABLE 1. EXPRESSIONS FOR THE ANGULAR FREQUENCY, ω, OF THE INERTIAL AND MAGNETIC MODES IN THREE CASES, $\gamma \ll 1$, $\gamma = 1$, $\gamma \gg 1$†

	inertial mode	magnetic mode
$\gamma \ll 1$	$\dfrac{-\beta k}{k^2+l^2}(1+\frac{1}{4}\gamma^2)$	$\frac{1}{2}\,\mathrm{sgn}\,(\beta)\,\gamma V_A\lvert\kappa\rvert$
	$\left[\dfrac{-2\,\mathrm{sgn}(\beta)\,V_A\lvert\kappa\rvert}{\gamma}(1+\frac{1}{4}\gamma^2)\right]$	$\left[\dfrac{\beta k\gamma^2}{4(k^2+l^2)}\right]$
$\gamma = 1$	$-1{\cdot}207\dfrac{\beta k}{k^2+l^2}$	$0{\cdot}414\,\mathrm{sgn}(\beta)\,V_A\lvert\kappa\rvert$
	$[-2{\cdot}414\,\mathrm{sgn}\,(\beta)\,V_A\lvert\kappa\rvert]$	$\left[0{\cdot}207\dfrac{\beta k}{k^2+l^2}\right]$
$\gamma \gg 1$	$\dfrac{-\beta k\gamma}{2(k^2+l^2)}\left(1+\dfrac{1}{\gamma}\right)$	$\mathrm{sgn}\,(\beta)\,V_A\lvert\kappa\rvert\,(1-\gamma^{-1})$
	$[-\mathrm{sgn}\,(\beta)\,V_A\lvert\kappa\rvert\,(1+\gamma^{-1})]$	$\left[\dfrac{\beta k\gamma}{2(k^2+l^2)}\left(1-\dfrac{1}{\gamma}\right)\right]$

† Equivalent expressions are given in square brackets. The parameter

$$\gamma \equiv \lvert 2V_A(k^2+l^2)\,(k\cos\theta+l\sin\theta)/k\beta\rvert = \lvert\kappa(k^2+l^2)/kk_0^2\rvert.$$

For given wavenumbers, k and l, and angle θ made by the impressed uniform magnetic field $\mathbf{B}_0$ to the west-east (x) direction, γ can be regarded either as a dimensionless measure of the Alfvén speed, V_A (proportional to B_0), in terms of β, the rate of variation of the Coriolis parameter with latitude, or alternatively as an inverse measure of β in terms of V_A, in which case γ has some of the characteristics of a Rossby number. If V_A and β are regarded as fixed γ varies only with k, l and θ, and when l and θ are also regarded as fixed, γ is a measure of k, the east–west wavenumber. This measure, though complicated in general, reduces to k proportional to $\gamma^{\frac{1}{2}}$ when $\theta = l = 0$ (see table 2 and figure 4).

When $\gamma \gg 1$ (see table 1) the magnetic field strength is so high that the frequencies of both modes have very nearly the same magnitude, being equal to $V_A\lvert\kappa\rvert$ multiplied by factors differing from unity by only γ^{-1}. In these circumstances both modes are virtually non-dispersive, corresponding to ordinary hydromagnetic waves propagating in both directions along the magnetic lines of force at the Alfvén speed, V_A.

At the other extreme, $\gamma \ll 1$, rotational effects are so strong that

$$\omega_i \doteqdot -2\,\mathrm{sgn}\,(\beta)\,V_A\lvert\kappa\rvert/\gamma = -\beta k/(k^2+l^2) \tag{5·35}$$

and

$$\omega_m \doteqdot \tfrac{1}{2}\gamma\,\mathrm{sgn}\,(\beta)\,V_A\lvert\kappa\rvert = V_A^2\kappa^2(k^2+l^2)/k\beta \tag{5·36}$$

(cf. equations (2·6) and (2·7)), both modes then being highly dispersive. According to equations (5·35) and (5·36), $\lvert\omega_i/\omega_m\rvert$ is then equal to $4/\gamma^2$, so that the inertial mode has a very much higher frequency than the magnetic mode. ω_i is virtually independent of V_A and differs only slightly from the Rossby–Haurwitz value (see equation (4·14)); in magnitude it greatly exceeds $V_A|\kappa|$, by a factor $2/\gamma$. The frequency of the magnetic mode depends strongly on both V_A and β; in magnitude it is much less than $V_A|\kappa|$, by a factor $\frac{1}{2}\gamma$.

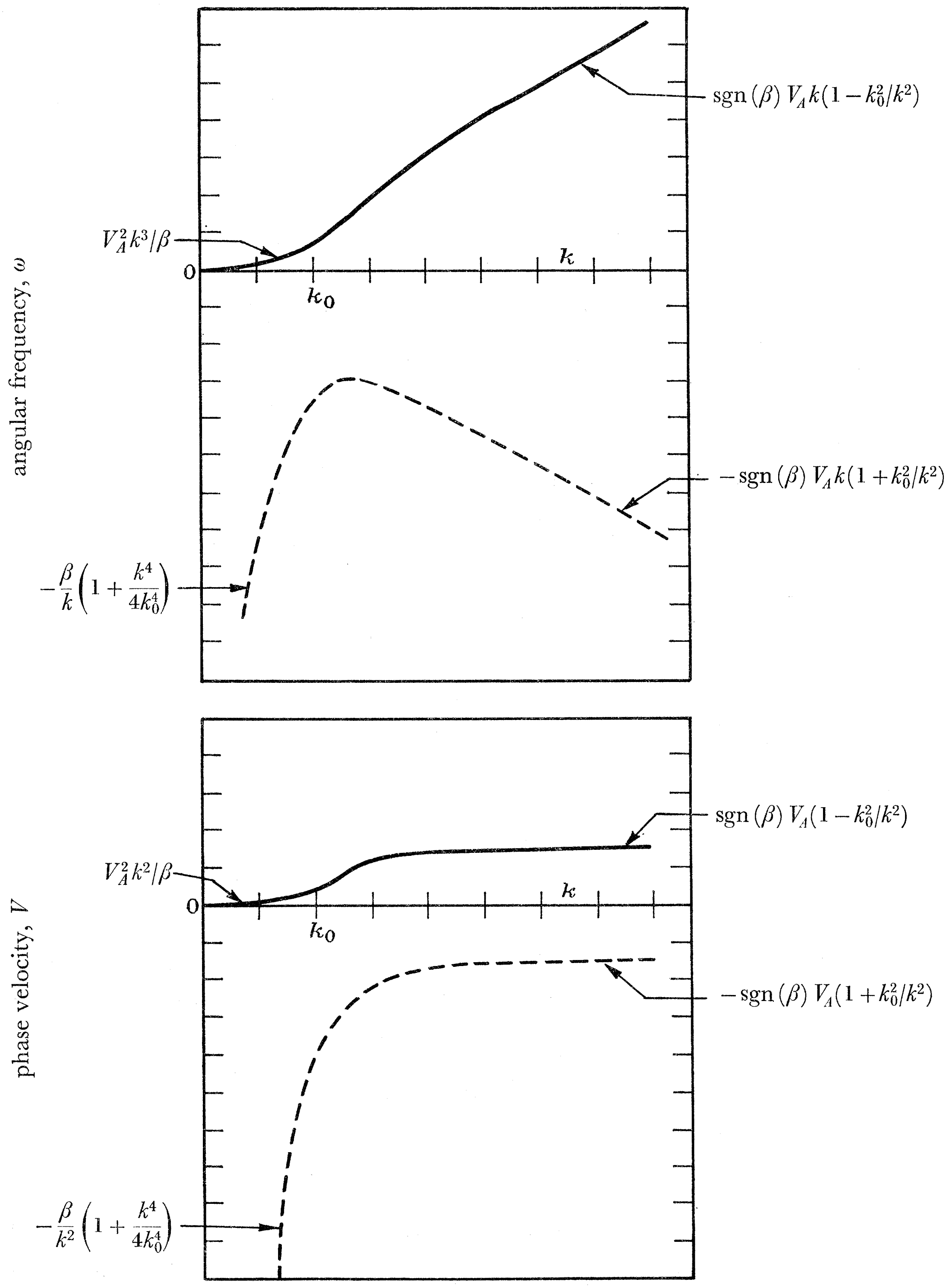

FIGURE 4. For legend see facing page.

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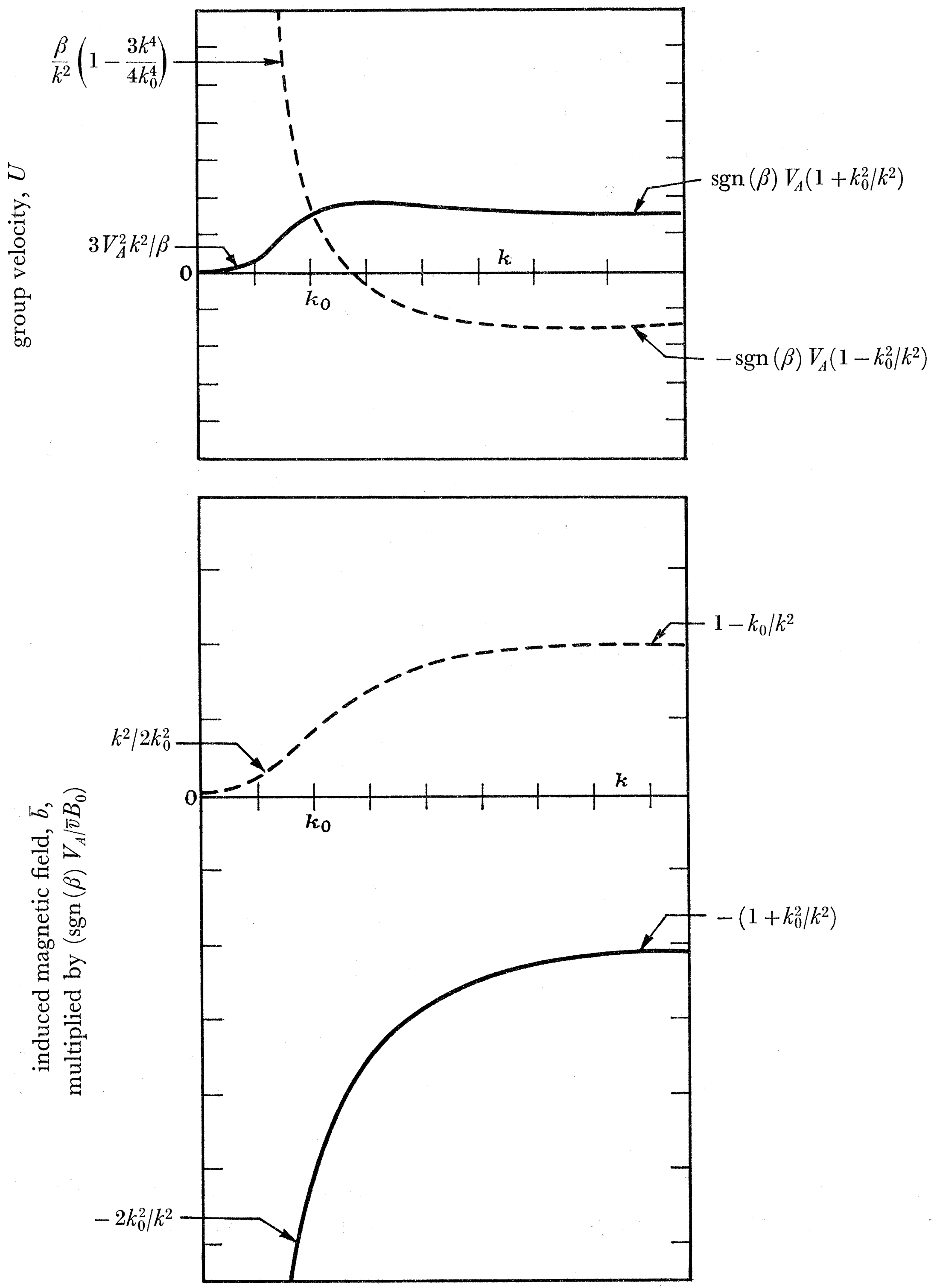

FIGURE 4. Angular frequency, phase velocity, group velocity and induced magnetic field as a function of wavenumber in the x direction when $l=0$, $\theta=0$. The expressions written alongside the curves give the asymptotic behaviour in the limits $k/k_0\to 0$ and $k/k_0\to\infty$; $k_0\equiv\sqrt{(|\beta|/2V_A)}$. ----, inertial mode; —— magnetic mode.

The speeds (relative to the basic flow) of phase propagation in the x direction, together with the corresponding group velocities, of the inertial and magnetic modes, respectively

$$V_i \equiv \omega_i/k, \quad V_m \equiv \omega_m/k, \quad U_i \equiv \partial\omega_i/\partial k, \quad U_m \equiv \partial\omega_m/\partial k, \tag{5·37}$$

are readily derived from equations (5·31) or (5·32). A useful relation, obtained by differentiating equation (5·27) with respect to k, is

$$\begin{aligned}\frac{\partial\omega}{\partial k} &= \left[\frac{\beta(k^2-l^2)}{(k^2+l^2)^2} + \frac{2V_A^2\kappa\cos\theta}{\omega}\right] \Big/ \left(1+\frac{V_A^2\kappa^2}{\omega^2}\right) \\ &= V_A\left[\frac{V_A\kappa\cos\theta}{\omega} + \frac{\operatorname{sgn}(\beta)}{\gamma}\left\{\frac{|\kappa|}{\gamma}\frac{\partial\gamma}{\partial k} - |\cos\theta|\right\}\right] \Big/ \left(1+\frac{\operatorname{sgn}(\beta)\,V_A|\kappa|}{\omega\gamma}\right).\end{aligned} \tag{5·38}$$

(d) *Modes for which $l = \theta = 0$*

Before concluding this section it will be instructive to consider the particularly simple case of modes having no dependence on y (i.e. $l = 0$, see equation (5·20)), when the basic magnetic field is parallel to the x axis (i.e. $\theta = 0$, see equation (5·24)), in which case

$$\left.\begin{aligned}\kappa = k, \quad \bar{u} = 0, \quad \bar{v} = -\mathrm{i}\bar{\zeta}/k, \quad \bar{b}_x = 0, \\ \bar{b}_y = \mathrm{i}B_0\bar{\zeta}/\omega, \quad \omega^2 + \beta\omega/k - V_A^2k^2 = 0\end{aligned}\right\} \tag{5·39}$$

(cf. equations (5·24), (5·21), (5·22) and (5·23)). The values of γ and ω are as given by equations (5·30) and (5·31) or (5·32), the corresponding phase and group velocities (see equations (5·37) and (5·38)) being

$$\begin{aligned}V_i, V_m &= -\operatorname{sgn}(\beta)\,V_A\left[\pm\left(1+\frac{k_0^4}{k^4}\right)^{\frac{1}{2}} + \frac{k_0^2}{k^2}\right] \\ &= -\frac{\beta}{2k^2}\left[1 \pm \left(1+\frac{k^4}{k_0^4}\right)^{\frac{1}{2}}\right],\end{aligned} \tag{5·40}$$

and

$$\begin{aligned}U_i, U_m &= \operatorname{sgn}(\beta)\,V_A\left[\mp\frac{(1-k_0^4/k^4)}{(1+k_0^4/k^4)^{\frac{1}{2}}} + \frac{k_0^2}{k^2}\right] \\ &= \frac{\beta}{2k^2}\left[1 \pm \frac{(1-k^4/k_0^4)}{(1+k^4/k_0^4)^{\frac{1}{2}}}\right].\end{aligned} \tag{5·41}$$

By the fifth of equations (5·39), corresponding to the two roots ω_i and ω_m of the sixth equation (see equations (5·31) or (5·32)), there are two values of $\bar{b}_y$, namely $\bar{b}_i$ and $\bar{b}_m$ (say), where

$$\bar{b}_i = -B_0\bar{v}/V_i, \quad \bar{b}_m = -B_0\bar{v}/V_m. \tag{5·42}$$

Figure 4 illustrates graphically the expressions for ω, V, U and $\bar{b}$ given by equations (5·31) or (5·32), (5·40), (5·41) and (5·42). When $k \ll k_0$ (but $k \neq 0$ since we are discussing the case $k \gg l$, $l = 0$),

$$\left.\begin{aligned}&\omega_i = -\beta(1+k^4/4k_0^4)/k, && \omega_m = V_A^2k^3/\beta, \\ &V_i = -\beta(1+k^4/4k_0^4)/k^2, && V_m = V_A^2k^2/\beta, \\ &U_i = \beta(1-3k^4/4k_0^4)/k^2, && U_m = 3V_A^2k^2/\beta, \\ &\bar{b}_i = \operatorname{sgn}(\beta)\,\frac{B_0\bar{v}\,k^2}{2V_A\,k_0^2}\left(1-\frac{k^4}{k_0^4}\right), && \bar{b}_m = -\operatorname{sgn}(\beta)\,\frac{2B_0\bar{v}k_0^2}{V_Ak^2}.\end{aligned}\right\} \tag{5·43}$$

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At the other extreme, $k \gg k_0$,

$$\left.\begin{aligned} \omega_i &= -\operatorname{sgn}(\beta)\, V_A k(1+k_0^2/k^2), & \omega_m &= \operatorname{sgn}(\beta)\, V_A k(1-k_0^2/k^2),\\ V_i &= -\operatorname{sgn}(\beta)\, V_A(1+k_0^2/k^2), & V_m &= \operatorname{sgn}(\beta)\, V_A(1-k_0^2/k^2),\\ U_i &= -\operatorname{sgn}(\beta)\, V_A(1-k_0^2/k^2), & U_m &= \operatorname{sgn}(\beta)\, V_A(1+k_0^2/k^2),\\ \bar{b}_i &= \operatorname{sgn}(\beta)\frac{B_0\bar{v}}{V_A}\left(1-\frac{k_0^2}{k^2}\right), & \bar{b}_m &= -\operatorname{sgn}(\beta)\frac{B_0\bar{v}}{V_A}\left(1+\frac{k_0^2}{k^2}\right). \end{aligned}\right\} \tag{5·44}$$

When $k_0 = k$, $U_i = U_m = \operatorname{sign}(\beta)\, V_A$ (see equation (5·41)). The corresponding values of the other quantities are then as follows:

$$\left.\begin{aligned} &\omega_i = -1{\cdot}07\beta/k_0, & &V_i = -1{\cdot}07\beta/k_0^2 = -2{\cdot}14\operatorname{sgn}(\beta)\, V_A,\\ &\bar{b}_i = 0{\cdot}47\operatorname{sgn}(\beta)\, B_0\bar{v}/V_A; & &\omega_m = 0{\cdot}21\beta/k_0,\\ &V_m = 0{\cdot}21\beta/k_0^2 = 0{\cdot}42\operatorname{sgn}(\beta)\, V_A, & &\bar{b}_m = -2{\cdot}4\operatorname{sgn}(\beta)\, B_0\bar{v}/V_A. \end{aligned}\right\} \tag{5·45}$$

When $k = (3+2\sqrt{3})^{\frac{1}{4}}\, k_0 = 1{\cdot}529 k_0$, $|U_m|$ has its maximum value, equal to $1{\cdot}18 V_A$.

The term $\operatorname{sgn}(\beta)\, U_i$ is positive or negative according as $k \lessgtr 1{\cdot}316 k_0$. When $k = 1{\cdot}316 k_0$, $|\omega_i|$ has its maximum value and U_i vanishes; the corresponding values of the other quantities then are

$$\left.\begin{aligned} &\omega_i = -1{\cdot}14\beta/k_0, \quad V_i = -0{\cdot}87\beta/k_0^2 = -1{\cdot}74\operatorname{sgn}(\beta)\, V_A,\\ &\bar{b}_i = 0{\cdot}57\operatorname{sgn}(\beta)\, B_0\bar{v}/V_A; \quad \omega_m = 0{\cdot}76\operatorname{sgn}(\beta)\, V_A k_0,\\ &V_m = 0{\cdot}58\operatorname{sgn}(\beta)\, V_A, \quad U_m = 1{\cdot}15\operatorname{sgn}(\beta)\, V_A, \quad \bar{b}_m = -1{\cdot}73\operatorname{sgn}(\beta)\, B_0\bar{v}/V_A. \end{aligned}\right\} \tag{5·46}$$

Comparison of the last pair of equations (5·43) with the last pair of equations (5·44) brings out one important effect of rotation. In the latter case, i.e. $k \gg k_0$, both modes become ordinary hydromagnetic waves, characterized by equipartition between magnetic and kinetic energy. In contrast, when $k \ll k_0$, the former case, the magnetic mode possesses much more magnetic energy than kinetic energy, and vice versa for the inertial mode.

6. Hydromagnetic oscillations of the Earth's core

(a) *Spherical harmonic coefficients*

Let us now apply the results of § 5 to the Earth's core, with the proviso that until more realistic theoretical models have been analysed any conclusions to be drawn from the application of the β-plane model must be regarded as tentative.

It will be convenient to introduce the coefficients m and n that arise in the spherical harmonic analysis of the geomagnetic field. Thus, that part of the magnetic field at the Earth's surface which originates within the Earth is expressed in terms of a potential function:

$$\Phi \equiv R_s \sum_{n=1}^{\infty} \sum_{m=0}^{n} P_n^m(\sin\phi)\,(R_s/R')^{n+1}\{g_n^m \cos m\lambda + h_n^m \sin m\lambda\}, \tag{6·1}$$

where R_s is the radius of the Earth, $(R', \frac{1}{2}\pi - \phi, \lambda)$ are spherical polar coordinates, $P_n^m(\sin\phi)$ are the Schmidt quasi-normalized polynomials of argument $\sin\phi$, and g_n^m and h_n^m are amplitude coefficients (see Chapman & Bartels 1940). The wavenumbers k and l introduced in § 4 (see equations (4·10) and (5·20)) are related to m and n as follows:

$$m = k(R+H)\cos\phi_1, \quad n-m = l(R+H), \tag{6·2}$$

where $(R+H)$ is the radius of the outer boundary of the liquid core (see figure 3). We can define a spherical harmonic coefficient m_0 corresponding to the quantity k_0 (see equation (5·25)) as follows:

$$m_0 \equiv k_0(R+H)\cos\phi_1 = (|\beta|/2V_A)^{\frac{1}{2}}(R+H)\cos\phi_1, \tag{6·3}$$

and if we take $2\Omega = 1{\cdot}4\times10^{-4}$ rad/s, $(R+H) = 3{\cdot}47\times10^8$ cm, $\phi_1 = 34°$ (see the discussion following equation (3·6)), $V_A(\text{cm/s}) = 0{\cdot}091B_0$ (corresponding to a mean core density of 11 g/cm^3), then

$$\left.\begin{aligned} k &= 3{\cdot}5\times10^{-9}m(\text{cm}^{-1}), & l &= 2{\cdot}9\times10^{-9}(n-m)\,(\text{cm}^{-1}),\\ k_0 &= 1{\cdot}3\times10^{-6}B_0^{-\frac{1}{2}}(\text{cm}^{-1}), & m_0 &= 3{\cdot}7\times10^2B_0^{-\frac{1}{2}}, \end{aligned}\right\} \tag{6·4}$$

where B_0 is measured in oersted (see equations (6·2), (6·3), (5·25), (3·6) and (2·1)).

Unless B_0 for the core greatly exceeds 500 Oe, m_0 is certainly greater than 17. When the basic magnetic field is parallel to latitude circles (i.e. $\theta = 0$ or π), then $|\kappa| = k$ (see equation (5·24)), so that by equation (5·26) we have the two cases $\gamma \gg 1$ and $\gamma \ll 1$ (see table 1) according as $k_0^2 \ll (k^2+l^2)$ or $k_0^2 \gg (k^2+l^2)$. In terms of the spherical harmonic coefficients these criteria become

$$m_0^2 \ll (m^2+(n-m)^2\cos^2\phi_1) \quad \text{and} \quad m_0^2 \gg (m^2+(n-m)^2\cos^2\phi_1).$$

As the description of the g.s.v. in terms of spherical harmonics hardly warrants the use of high-order components, with $m_0 > 17$ attention can be concentrated on the latter case. Then

$$\omega_i = -\frac{\beta k}{k^2+l^2} = \frac{2\Omega m\cos^2\phi_1}{m^2+(n-m)^2\cos^2\phi_1}, \tag{6·5}$$

$$\omega_m = \frac{V_A^2k(k^2+l^2)}{\beta} = -\frac{V_A^2m(m^2+(n-m)^2\cos^2\phi_1)}{2\Omega(R+H)^2\cos^4\phi_1}, \tag{6·6}$$

$$V_i = \frac{2\Omega(R+H)\cos^3\phi_1}{m^2+(n-m)^2\cos^2\phi_1}, \tag{6·7}$$

$$V_m = -\frac{V_A^2(m^2+(n-m)^2\cos^2\phi_1)}{2\Omega(R+H)\cos^3\phi_1}, \tag{6·8}$$

$$U_i = -\frac{2\Omega(R+H)\cos^3\phi_1(m^2-(n-m)^2\cos^2\phi_1)}{(m^2+(n-m)^2\cos^2\phi_1)^2}, \tag{6·9}$$

and

$$U_m = \frac{V_A^2(3k^2+l^2)}{\beta} = -\frac{V_A(3m^2+(n-m)^2\cos^2\phi_1)}{2\Omega(R+H)\cos^3\phi_1} \tag{6.10}$$

(see equations (6·2), cf. equations (5·37), (5·38) and (5·43)).

Characteristic periods of the inertial modes are several days (see equation (6·5)). Although the electrical conductivity of the mantle is much less than that of the core, if the usual estimates of mantle conductivity (see Tozer 1959) are acceptable, magnetic variations in the core on a time scale less than a few years cannot penetrate to the Earth's surface. Hence we do not expect the inertial mode to manifest itself in the magnetic record, although it would be interesting to consider whether such short-period oscillations could have some detectable geophysical effects. In this connexion, note that there is a nearly diurnal nutation of the Earth whose properties might be influenced by core motions (see Vicente & Jeffreys 1964).

It would also be interesting to seek evidence for the inertial mode on the Sun, or on the planet Jupiter, whose magnetic field may be the result of motions in the lower part of his atmosphere (Hide 1966*a*, *b*).

(*b*) *The magnetic mode in the core*

The magnetic mode is much slower than the inertial mode. Table 2 gives values of the oscillation period, $2\pi/\omega_m$, the westward phase velocity, $-V_m$, relative to the basic flow, and the dispersion time, $T_m \equiv 2\pi/k\,|U_m - V_m|$, based on equations (6·4), (6·6), (6·8) and (6·10) (see also equation (4·12)) for $m = 1$ to 5 in the case $n = m$ (i.e. $l = 0$, see equation (6·2)) and $\theta = 0$ (see §5, especially equations (5·24) and (5·43)). Because the strength, B_0, of the toroidal field in the core is not known (see §2) the calculations summarized in table 2 were carried out for four different values, namely 50, 100, 200 and 500 Oe.

TABLE 2. SOME PROPERTIES OF FREE HYDROMAGNETIC OSCILLATIONS OF THE EARTH'S CORE CALCULATED FOR DIFFERENT ASSUMED VALUES OF THE STRENGTH OF THE BASIC TOROIDAL MAGNETIC FIELD†

sectoral harmonic coefficient, $m = n$ ...	1	2	3	4	5	B_0 (Oe)
oscillation period, $2\pi/\omega_m$ (y)	1.2×10^4	1.5×10^3	4.3×10^2	1.8×10^2	9.3×10	50
	2.9×10^3	3.6×10^2	1.1×10^2	4.5×10	2.3×10	100
	7.2×10^2	9.0×10	2.7×10	1.2×10	5·8	200
	1.2×10^2	1.5×10	4·3	1·8	9.3×10^{-1}	500
westward phase speed, $-V_m$ (mm/s)	7.5×10^{-3}	3.0×10^{-2}	6.7×10^{-2}	1.2×10^{-1}	1.9×10^{-1}	50
	3.0×10^{-2}	1.2×10^{-1}	2.7×10^{-1}	4.8×10^{-1}	7.5×10^{-1}	100
	1.2×10^{-1}	4.8×10^{-1}	1·1	1·9	3·0	200
	7.5×10^{-1}	3·0	6·7	1.2×10	1.9×10	500
dispersion time, T_m (y)	5.8×10^3	7.2×10^2	2.2×10^2	9.1×10	4.7×10	50
	1.5×10^3	1.8×10^2	5.4×10	2.3×10	1.2×10	100
	3.6×10^2	4.5×10	1.4×10	6·0	2·9	200
	5.8×10	7·2	2·2	9.1×10^{-1}	4.7×10^{-1}	500

† The properties of the magnetic mode only are given, calculated for the special case of no variation with latitude (i.e. $n = m$, where n and m are the usual spherical harmonic coefficients), when the basic magnetic field, strength B_0, has no meridional component (i.e. $\theta = 0$). The dispersion time $T_m \equiv 2\pi/k\,|V_m - U_m|$, where U_m is the group velocity, also directed westward. (In magnitude the corresponding inertial modes have much shorter periods $(2\pi/\omega_i)$ and dispersion times $(T_i \equiv 2\pi/k|V_i - U_i|)$, of the order of days, and much higher phase and group velocities (V_i and U_i respectively), of the order of 10^2 m/s, than the magnetic modes. $V_i > 0$ and $U_i < 0$, corresponding to eastward phase propagation and westward group propagation for the inertial mode.)

(*c*) *Excitation of hydromagnetic oscillations*

Irrespective of the details of the energy source responsible for core motions, one can safely conjecture that whilst these motions might exhibit some degree of irregularity, they are by no means completely random. As was shown in §2, the Earth's rotation probably renders the largest-scale motions highly anisotropic.

Until knowledge of the spectrum of core motions has much improved and the theory of hydromagnetic turbulence has advanced beyond its present rudimentary state, any proposed mechanism for exciting free hydromagnetic oscillations must be regarded as speculative. A few general remarks might, however, be in order.

In isotropic three-dimensional turbulence, non-linear interactions between different scales of motion give rise to a cascade of energy from the large eddies to the small eddies, the largest eddy in the system being energetically potent (i.e. capable of converting the energy

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of the source into kinetic energy of hydrodynamical motion). In contrast, in highly anisotropic flows, exemplified by two-dimensional turbulence, the transfer of energy from an eddy of any given size to a smaller eddy must be accompanied by the simultaneous passage of energy to larger eddies (Onsager 1949; Batchelor 1956; Lorenz 1953; Fjørtoft 1953). In such systems eddies on all scales up to the size of the container should be present, even if the energetically potent eddies are small compared to the size of the container.

As core motions are probably highly anisotropic, the size of the largest eddies should, irrespective of the size of the energetically potent eddies (which could be quite small), be comparable with the diameter of the core. The general features of the geomagnetic secular variation, with its regions of rapid change covering areas of continental size (see § 7), suggest that this is the case.

Owing to the rapid rotation of the Earth, quite shallow topographic features of the bounding surfaces of the liquid core might influence hydrodynamical motions at all levels in the core (see § 2 above), and thus have a marked effect on the spectrum of free hydromagnetic oscillations. In the absence of friction the most effective topographic features would have the largest horizontal scale. However, the effects of the largest features, including, presumably, those of the axisymmetric bulges, are more strongly opposed by friction than are those due to smaller-scale features (see Hide 1961, 1963).

The relationship between the spectrum of free hydromagnetic oscillations in the core and the geomagnetic field observed at the surface of the Earth depends on the filtering action of the weakly conducting mantle. This action suppresses the high-frequency components (i.e. those whose periods are less than a few years) and attenuates the lower-frequency components by an amount which depends not only on the oscillation period, but also on the distribution of electrical conductivity in the mantle, especially on its radial variation, which has not yet been fully elucidated (see Tozer 1959).

The relative amplitudes of the inertial and magnetic modes of oscillation will depend on the nature of the excitation mechanism. If these amplitudes are comparable with one another, one consequence of the complete dissipation in the lower mantle of the electric currents associated with oscillations of the poloidal magnetic field, $\mathbf{B}_p$, at the inertial mode frequency might be an enhanced electromagnetic coupling of the motion of the core to that of the surrounding mantle. Whether or not this enhanced coupling has any practical significance in the discussion of variations of the angular velocity of rotation of the mantle remains to be evaluated (see § 8 below).

The next step will be to examine the evidence for free hydromagnetic oscillations in the core by comparing the predictions of the preceding theoretical discussion with the observed properties of the geomagnetic secular variation.

7. The geomagnetic secular variation (g.s.v.)

(a) *Properties of the g.s.v.*

Direct measurements of the geomagnetic elements go back only a century or so. They have been made at both permanent and temporary observatories, most of which are located in the populated regions of the Earth. Consequently, secure and detailed knowledge of the behaviour of the geomagnetic field is available over only a limited region of space, for a very short (geologically speaking) period of time (see Vestine, Laporte, Cooper, Lange & Hendrix 1947*a*; Vestine, Laporte, Lange & Scott 1947*b*). Thanks, however, to recent archaeomagnetic studies of human artifacts, such as hearths, pottery and kilns, and to palaeomagnetic studies of both sedimentary and igneous rocks, we are not completely ignorant of the history of the geomagnetic field (see Nagata 1953; Runcorn 1955*a*, 1956; Blackett 1956; Gaibar-Puertas 1953; Cook & Belshé 1958; Aitken, Harold & Weaver 1964; Irving 1959, 1964).

The main properties of the geomagnetic secular variation have been discussed elsewhere (see, for example, recent reviews by Hide & Roberts 1961; Yukutake 1962; Jacobs 1963). Observatory data reveal the following:

(i) It is the non-dipole field that undergoes the most rapid secular changes, the 'centre of gravity' of the spherical harmonic series for the secular variation field lying between $n = 3$ and $n = 4$ (see Runcorn 1955*b*). The r.m.s. rate of change of the non-dipole field amounts to $50\gamma/\text{y}$ ($1\gamma = 10^{-5}\,\text{Oe} = 10^{-9}\,\text{Wb/m}^2$), and the maximum rates of change (regardless of sign) are about $150\gamma/\text{y}$. $50\gamma/\text{y}$ represents a proportionate rate of change of 2·5% per year for the non-dipole field, suggesting a time scale of about 40 y for the non-dipole component of the g.s.v. (The dipole field of the Earth has, over the past century, changed at an average rate of $35\gamma/\text{y}$ (i.e., 0·1 %/y), suggesting a time scale of about 1000 y for the dipole component of the g.s.v.).

(ii) On typical magnetic maps, lines of equal annual change of any element (isopors), at any epoch, form a series of sets of oval curves surrounding points at which the changes are most rapid (isoporic foci). The sets of isopors cover areas of continental size and are separated by regions over which the changes are small.

(iii) The isoporic foci migrate slowly westward at a fraction of a degree of longitude per year (equivalent to a fraction of a millimetre per second), as does the non-dipole field; there is evidence that this has been occurring for the past 300 years.

(iv) In addition to the westward drift, the pattern of the secular variation field undergoes perceptible alterations in form in only a few decades.

(v) Whilst the whole pattern of the g.s.v. shows no striking detailed correlation with topographic features of the Earth's surface, it is noteworthy that over the Pacific hemisphere (the region lying between 120° E and 80° W) the g.s.v. is systematically weaker, by a factor of three, than over the rest of the Earth's surface.

The westward drift. The westward drift of the geomagnetic field, property (iii) above, is a particularly striking feature, to which a certain amount of recent attention has been given (see Nagata 1962). Analyses have dealt with three aspects of the phenomenon, namely (*a*) the dependence of the westward drift rate on latitude, (*b*) the possibility that each spherical harmonic component has its own characteristic drift rate, and (*c*) the extent to which the secular variation can be accounted for in terms of a uniform drift of a fixed field pattern.

Uncertainties in the analyses, especially those stemming from property (iv) of the g.s.v. (see above), preclude any definite statement being made in regard to (*a*). The upshot of (*b*) is that although entirely reliable estimates of the individual westward drift rates of all major spherical harmonic components of the field have not yet been obtained, there is no doubt that the rate of drift of the equatorial dipole during the past hundred years or so has been less than one-third that of the non-dipole field, the latter having moved at about 0·2° long./y (see table 3 below). As regards (*c*), in spite of a number of uncertainties and ambiguities, over sufficiently short periods of time (no more than a decade or so) a uniform westward drift of the observed non-dipole field can, evidently, account for a substantial fraction of the observed geomagnetic secular variation.

Table 3. Westward drift of spherical harmonic components

(Units: degrees of longitude per year)

	spherical harmonic coefficients										
n ...	1	2	2	3	3	3	4	4	4	4	mean
m ...	1	1	2	1	2	3	1	2	3	4	$n \geqslant 2$
epoch (reference)											
1829–1885 (1)	0·11?	0·26?	0·80?	—	—	—	—	—	—	—	0·53?
1829–1945 (2)	0·062?	0·270 ±0·016	0·341 ±0·018	0·113?	0·037	0·234 ±0·024	—	—	—	—	0·199
1907·5–1945 (2)	0·003	0·235	0·363	−0·080	−0·080	0·243	—	—	—	—	0·136
1922 (3)	0·01	0·13	0·47	−0·03	−0·04	0·38	—	—	—	—	0·18
1945 (3)	0·02	0·29	0·23	0·03	−0·08	0·22	—	—	—	—	0·14
1955 (4)	0·01	0·23	0·26	0·00	−0·15	0·17	—	—	—	—	0·10
1955–1960 (5)	0·06	0·26?	0·32	0·35	−0·08	0·14	0·19?	0·10?	0·17	0·14	0·18
1960 (6)	0·06	0·20	0·32	0·11	−0·07	0·20	0·13	0·12	0·15	0·17	0·15
1965 (7)	0·07	0·16	0·33	0·14	−0·04	0·18	−0·19	0·09	0·13	0·16	0·11
mean	0·05	0·23	0·38	0·08	−0·06	0·22	0·04	0·10	0·15	0·16	0·19

(1) Carlheim-Gyllensköld (1896); (2) Bullard *et al.* (1950); (3) Whitham (1958); (4) Yukutake (1964, private communication) based on analyses by Finch & Leaton (1957) and Nagata & Rikitake (1957); (5) Nagata (1962), see also Yukutake (1962); (6) Cain *et al.* (1964), see also Daniels & Cain (1964); (7) Leaton & Evans (1964).

Table 3 represents an attempt to summarize the results of several investigations of the westward drift rate of different spherical harmonic components (Carlheim-Gyllensköld 1896; Bullard, Freedman, Gellman & Nixon 1950; Whitham 1958; Nagata & Rikitake 1957; Nagata 1962; Yukutake 1962, 1964 (private communication); Cain, Daniels & Jensen 1964; Leaton & Evans 1964, see also Finch & Leaton 1957; Daniels & Cain 1964). For details of sources of geomagnetic data and methods of analysis employed in these investigations, together with occasional attempts at theoretical or heuristic interpretation, reference is made to the original papers (see, however, § 8 below).

Archaeomagnetic and palaeomagnetic data. Archaeomagnetic and palaeomagnetic data throw a little extra light on the properties of the g.s.v.

In regard to the drift of the geomagnetic field (property (iii)), several recent studies might be mentioned. Thus Aitken *et al.* (1964) have adduced evidence from a study of the magnetic properties of ancient pottery kilns in Britain that the motion of the equatorial dipole may have been eastward (relative to the mantle) during the period A.D. 900 to A.D. 1350. Brynjolfsson (private communication), from measurements of Icelandic lava flows, concludes that 'the higher harmonics (periods of the order of 300 to 500 y) drift westward while lower harmonics (periods of the order of 4000 to 5000 y) drift eastward' (see Brynjolfsson 1957). Related Japanese work (Kawai 1964; Kawai 1964 (private communication)) is in general agreement with that of Aitken *et al.* and of Brynjolfsson. Evidently the drift of the geomagnetic field may be an even more complex phenomenon than observatory data, taken alone, suggest, and deserving, therefore, of much further observational study.

Cox & Doell (1964), in their studies of the magnetization of lava flows in Alaska, the western United States, Hawaii and the Galapagos Islands, have found evidence that the secular variation has been systematically weaker over the Pacific hemisphere than over the rest of the Earth's surface since over 1 My ago, suggesting that property (v) revealed by observatory data is not a transient phenomenon.

(*b*) *Comparison of theory with observation*

Having outlined the main properties of the g.s.v., let us now compare them with those expected on the hypothesis that free hydromagnetic oscillations of the core (see § 6 above) contribute significantly to the phenomenon.

The mean of the nine values of the rate of westward drift of the equatorial dipole component ($n = 1, m = 1$) given in table 3 is 0·05° long./y, while that of the twenty corresponding values of the drift of the non-dipole components ($n \geqslant 2$) having $m = n$ is 0·28° long./y. These angular speeds correspond to linear speeds of 0·05 and 0·28 mm/s respectively in latitude $\phi_1 = 34°$ at a distance from the Earth's centre equal to the mean radius of the core. Taking the mean value of n for the Earth's non-dipole field as between 3 and 4, inspection of table 2 shows that the drift rates just calculated could be consistent with the hypothesis that the westward drift of the geomagnetic field is mainly a manifestation of the drift, relative to the core material, of the magnetic mode of free hydromagnetic oscillations of the core, provided that B_0 is about 100 Oe. The corresponding oscillation periods and dispersion times, also given in table 2, range from a decade or so for $m = 5$ and $m = 4$ to centuries and more for $m = 2$ and $m = 1$; these are comparable in magnitude with the time scales involved in the secular variation (see above).

At $B_0 = 50$ Oe the drift rates are a little low and the oscillation periods and dispersion times rather long; at 200 Oe the drift rates are distinctly high and the oscillation periods and dispersion times rather short; at 500 Oe the drift rates are much too rapid and the oscillation periods and dispersion times very much shorter than those found in the geomagnetic secular variation.

As was shown in § 2 (see equations (2·11) and (2·12)), a toroidal magnetic field, $\mathbf{B}_T$, of 100 Oe implies a magnetic Reynolds number, G, of about 20, and an average speed of zonal flow, $\langle \bar{V} \rangle$, certainly not greater than 0·03 mm/s and possibly an order of magnitude less. While this speed could be comparable with that of the drift of the equatorial dipole, it is a good deal less than that of the non-dipole field. Hence, the contribution of material motions in the core to the westward drift may be quite small and noticeable only in the case of the equatorial dipole.

The palaeomagnetic and archaeomagnetic results cited above—namely, that the equatorial dipole may, on occasions, have drifted slowly eastward relative to the mantle even when the non-dipole field drifted westward—are readily accounted for on the present hypothesis if an average eastward material motion of about 0·03 mm/s can occur in the core.

In principle, further tests of the hypothesis would be: (*a*) to examine in detail whether or not oscillation periods, phase velocities and group velocities depending qualitatively on m and n in the manner predicted by a rigorous theory of free hydromagnetic oscillations in the core are present in the spectrum of the geomagnetic secular variation (cf. equations (6·6), (6·8) and (6·10)); and (*b*) to seek evidence for the rapid eastward-propagating inertial modes associated with the slow westward-propagating magnetic modes in the core.

As regards (*a*), the details of the variation of drift rate with n and m as given in table 3 do not agree too well with equation (6·8). This lack of agreement may be more apparent than real, considering that we have compared the observed properties of a very complex phenomenon with those predicted by means of a highly simplified theoretical model. The discussion of more refined theoretical models, together with data analyses specifically devised to test the theoretical results, are needed to settle this point. As regards (*b*), the writer is not competent at present to add to the remarks given above, in the paragraph following equation (6·10) and at the end of § 6.

In spite of obvious uncertainties, the foregoing discussion suggests that some of the principal characteristics of the g.s.v. might be manifestations of free oscillations of the Earth's core and that, if they are, the strength of the toroidal magnetic field in the core is about 100 Oe. The discussion does not support an argument of Rikitake's (1956), asserting that the strength of the magnetic field in the core must exceed 10^5 Oe for free hydromagnetic oscillations to make a significant contribution to the geomagnetic secular variation (see § 1 above). This apparent contradiction can be resolved by noting that, owing to the mathematical complexity of his model, the only modes Rikitake was able to discuss quantitatively were those possessing spatial variations with respect to longitude only, in the case of a basic magnetic field possessing no azimuthal component—in our notation, he examined the case $l = 0$ and $\theta = 90°$. But in these circumstances, $\kappa = 0$ (see equation (5·24)) so that unless B_0 is infinite (see equation (2·1)) it does not enter the dispersion equation (5·23). Therefore, the modes considered by Rikitake are virtually unaffected by the magnetic field and hence must be regarded as atypical. Presumably the difference between Rikitake's 10^5 Oe and the infinite value of B_0 required on the present model to obtain frequencies of the right order of magnitude when $\kappa = 0$ is attributable to the simplifications incorporated in the β-plane model of § 5 above, although this remains to be proved.

(*c*) *Secular variation over the Pacific hemisphere*

It is hard to judge the extent to which free hydromagnetic oscillations of the core can account for the phenomenon discovered by Cox & Doell (see above), namely that for the past million years the secular variation over the Pacific hemisphere may have been systematically lower than the world-wide average. Owing to the westward phase and group propagation, together with the rapid dispersion of the magnetic mode of free hydromagnetic oscillations in the core, the g.s.v. might be expected to have this property if the position of the source of the free oscillations is restricted in longitude to regions of the core just west of the Pacific hemisphere. (In this connexion, we recall an observation by Gaibar-Puertas (1953) that according to the movement of fictitious lines joining observatories at which the secular variation vanishes instantaneously, secular changes evidently arise under points south of Asia and west of India and propagate outward (see Hide & Roberts 1961).)

The nature of the source is conjectural; it is not inconceivable that a shallow topographical feature of the core-mantle interface may be responsible, having regard for the possibility that the hydrodynamical effects of such a feature, interacting with core motions in a manner reminiscent of mountain waves in the atmosphere, might penetrate deeply within the core (see §§ 2 and 6 above).

The last suggestion is obviously speculative and is not intended, therefore, to be taken seriously until it has been quantitatively assessed. It is included here simply because no satisfactory explanation of the low value of the secular variation over the Pacific seems to have been given. As Cox & Doell point out, the phenomenon cannot be accounted for by postulating higher electrical conductivity or magnetic permeability of the mantle under the Pacific than elsewhere, since the most rapid secular changes over the Pacific are not significantly slower than elsewhere.

8. Concluding remarks

In his discussion of the homogeneous dynamo mechanism Bullard worked out in some detail a theory of the westward drift of the geomagnetic field (see Bullard *et al.* 1950; Bullard & Gellman 1954). Assuming that during their motion individual fluid particles tend to conserve angular momentum, he argued that if meridional flow occurs in the core, the concomitant inward advective transfer of angular momentum would cause the inner parts of the core to rotate more rapidly than the outer parts. For such an angular velocity distribution to remain steady on the average, advective angular momentum transfer inward must be balanced by an outward transfer due to friction. Bullard showed that although viscosity would be inadequate to provide this frictional transfer, electromagnetic forces might suffice. Further, except at the outer extremity of the core, where viscosity demands that there be no motion relative to the mantle, electromagnetic coupling between the core and the mantle could allow the outer part of the core to move westward relative to the mantle. Hence, if secular variation sources are located mainly in the upper reaches of the core they would appear to drift westward relative to a fixed observer at the Earth's surface.

In addition to accounting roughly for one of the chief properties of the g.s.v., Bullard's theory of the westward drift provided a model of core motions which subsequently proved useful in the discussion of variations in the length of the day (see Bullard *et al.* 1950; Rochester 1960; Munk & MacDonald 1960; Vestine 1952, 1962; Roden 1963). Observe, however, that his assumption that individual fluid particles conserve angular momentum, though *a priori* reasonable, is not generally valid; there are laboratory systems in which it is known to break down under very definite conditions (see Hide 1958), it is not valid in the Earth's atmosphere except at low latitudes, and it cannot be true in the solar atmosphere, which rotates *faster* at the equator than near the poles.

When the assumption does break down, more complicated motions, characterized by variations with respect to λ, the angle of longitude, occur. The present state of theoretical hydrodynamics is too rudimentary to lead to much insight into the properties of such motions, though it is fairly plausible that the constituent eddies might drift horizontally, perhaps in a complicated way, at rates which are somewhat less than the motion within an individual eddy.

If the westward drift is a direct manifestation of zonal material motions in the core, the average speed, $\langle \bar{V} \rangle$ (see § 2 (c)), of these motions cannot be less than about 0·3 mm/s. On the other hand, if, as the present work suggests, the westward drift is a manifestation of a wave motion in the core, $\langle \overline{V} \rangle$ may not exceed about 0·05 mm/s. Such a low value of $\langle \overline{V} \rangle$ would raise difficulties in the theory of variations in the length of the day, since the angular momentum available in the core for changing the rotation of the mantle would then be less than has hitherto been supposed. Hence, the re-examination of the theory of angular momentum transfer between core and mantle might be a crucial step in assessing whether or not free hydromagnetic oscillations in the core are as important as the present paper suggests. In estimating the strength of the mechanical coupling, it may be necessary to take into account surface roughness (see §§ 2 (*c*) and 7 (*c*)) and the possibility that the inertial mode may contribute to the electromagnetic torques acting (see § 6(*c*)).

Finally, consider the amplitude of the observed secular changes (property *i*, § 7 above). On the model discussed in this paper, the g.s.v. is due largely to the re-arrangement (as opposed to the creation and destruction) of lines of force of the Earth's poloidal magnetic field by motions in the core (see Roberts & Scott 1965; cf. Allan & Bullard 1958). According to equation (5·10) horizontal particle motions of about 0·2 mm/s are implied by the occurrence of secular changes of 150γ/y. Particle motions of this magnitude could also account for occasional eastward drifts of some of the higher-order spherical harmonic coefficients (see table 3) and need not be inconsistent with *average* zonal particle motions of less than 0·05 mm/s.

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The theory discussed in §§ 3, 4 and 5, together with tentative applications to the Earth's core and to the solar convective zone, were carried out in 1962, when I had the good fortune to participate in the Geophysical Fluid Dynamics Summer Program, directed by Dr G. Veronis, at the Woods Hole Oceanographic Institution. My interest in the work revived during my brief participation in the 1964 Woods Hole Summer Program, director Dr W. V. R. Malkus.

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